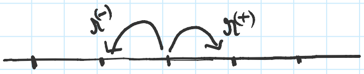
Assignment two

Monday, May 10, 2021 11:49 AM

What you will learn: Few important properties of Random walk, how to use method of generating function, a simple example of hydrodynamic limit, and a relation between RW and prime numbers.

Question 1:

Consider a discrete time random walk on a 1-d lattice.



$r^- + r^+ = 1$

(a) What is the probability $P_n(\ell)$ to be at site ℓ at nth time step, given that the walker started at the origin?

[Hint: Binomial distribution]

(b) What is the discrete time master equation for $P_n(\ell)$? Use this to to show that the characteristic function

$$G_n(k) = \sum_{\ell} e^{ik\ell} P_n(\ell) = \left(r^{(-)} e^{-ik} + r^{(+)} e^{ik}\right)^n \quad \text{---- (1)}$$

[You will need to use $P_0(\ell) = \delta_{\ell,0}$]

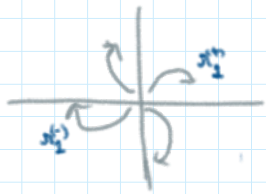
(c) Verify that the inverse fourier transform

$\frac{1}{2\pi}\int_{-\pi}^{\pi} dk\, G_n(k)\, e^{-ik\ell}$ gives the formula obtained in (a).

Question 2

Consider a generalization on d-dimensional grid $\mathbb{Z}^d$.

Jump prob along j-th direction are $\{r_j^{(-)}, r_j^{(+)}\}$



let $P_n(\bar{\ell} = \{\ell_1, \cdots, \ell_d\})$ is the probability for the walker to be at site $\bar{\ell}$ at step n, starting at the origin.

(a) Express $P_n(\bar{\ell})$ in terms of $P_n(\ell)$ in 1-d.

(b) Find the corresponding relation for $G_n(\bar{k})$ and $G_n(k)$ in 1-d.

(c) Use this to write an explicit formula for $G_n(\bar{k})$ and then show that

$$P_n(\bar{\ell}) = \frac{1}{(2\pi)^d} \int_{-\pi}^{\pi} dk_1 \cdots \int_{-\pi}^{\pi} dk_d\, e^{-i\bar{k}\cdot\bar{\ell}} \Bigg[\underbrace{\sum_{j=1}^{d} \left(r_j^- e^{-ik_j} + r_j^+ e^{ik_j}\right)}_{\gamma(\bar{k})\ [\text{define}]}\Bigg]^n . \quad \text{—— (2)}$$

(d) For the unbiased case $\left(r_j^{\pm} = \frac{1}{2d}\right)$, show that for large n limit

$$P_n(\bar{l}) \simeq \left(\frac{d}{2\pi n}\right)^{d/2} \exp\left[-\frac{d}{2n} l^2\right]$$

Taking continuous limit $n = t/a^2$ and $l = x/a$, with lattice spacing $a \to 0$, show that, the probability density to be at x at time t is

$$P_t(\bar{x}) \simeq \left(\frac{1}{4\pi D t}\right)^{d/2} \exp\left[-\frac{x^2}{4Dt}\right] \quad \text{with } D = \frac{1}{2d}$$

[You need to explain the correct prefactor]

[This is the distribution for a Brownian motion]

(e) The above demonstrates that, by suitably taking continuous limit (coarse graining) a random walk leads to the Brownian motion $\frac{\partial}{\partial t} P_t(\bar{x}) = D \nabla^2 P_t(\bar{x})$.

One can show the same by directly starting with the Master equation. Write the Master equation for the d-dimensional random walk defined above (Q2).

Show that, in the continuous limit $\partial_t P_t(\bar{x}) = \sum_{j=1}^{d} D_j \frac{\partial^2}{\partial x_j^2} P_t(\bar{x}) - v_j \frac{\partial}{\partial x_j} P_t(\bar{x})$

where $D_j = \frac{r_j^+ + r_j^-}{2}$ and $v_j = \frac{r_j^+ - r_j^-}{a}$ for $a \to 0$ limit.

This is a simple example of hydrodynamic limit of a stochastic process.

Question 3: Polya's theorem. Consider unbiased RW $\left(r^{\pm} = \frac{1}{2d}\right)$.

(a) We want to see, how frequently a RW returns to its starting point $\bar{l} = 0$.

For this, show that the mean number of times RW visit $\bar{l} = 0$ is $g(1,0)$, where

$$g(z, \bar{l}) = \sum_{n \geq 0}^{\infty} P_n(\bar{l}) \cdot z^n \quad \text{for } z < 1.$$

Here $g(z,\bar{l})$ is the generating function of $P_n(\bar{l})$.

Use the expression of $P_n(\bar{l})$ in Eq(2) and do the summation over n to write an integral formula for $g(z,\bar{l})$.

(b) In $d=1$, find an exact expression for $g(z,0)$ [You can use Mathematica]. Is $g(1,0)$ finite? What does this mean for the average number of visit to the origin?

In $d=2$, $g(z,0)$ diverge at $z \to 1$. Find the leading dependence of this divergence.

In $d=3$, find the numerical value of $g(1,0)$. What does it say for average number of visit to the origin?

Another quantity of interest is the probability of return to the origin. This gives an important result about recurrent nature of Random walk.

(c) To derive the probability of return, define

$$F_n(\bar{l}) := \text{probability that RW visit } l^{th} \text{ site for the first time at } n^{th} \text{ step } (n \geq 1).$$

Argue that the probability that RW comes back to the origin atleast once after it left the origin is $\sum_{n \geq 1} F_n(0)$.

(d) To evaluate this probability, first express

$P_n(\bar{l})$ in terms of $F_s(\bar{l})$ and $P_{n-s}(\bar{l})$ for $s \leq n$. [Hint: you need to use that for RW to be at $\bar{l}^{th}$ site at n^{th} step, it must have reached there once for the first time (say s) and then for the rest of the time $(n-s)$ it is an indep RW. You need to be careful for the $n=0$ step for which $P_0(\bar{l}) = \delta_{\bar{l},0}$.]

(e) These kind of relations are usually inverted using generating function method. Define

$$R(z,\bar{l}) = \sum_{n \geq 1} z^n F_n(\bar{l})$$

Then use the relation between $P_n(\bar{l})$ and $F_s(\bar{l})$ to show that

$$R(z,\bar{l}) = \begin{cases} 1 - \dfrac{1}{g(z,0)} & \text{for } \bar{l} = 0 \\[2ex] \dfrac{g(z,\bar{l})}{g(z,0)} & \text{for } \bar{l} \neq 0 \end{cases} \qquad (3)$$

where $g(z,\bar{l})$ is the generating function of $P_n(\bar{l})$ defined before in Q3a.

(f) Use the result in eq(3) to show that the probability to return to the origin is $1 - 1/g(1,0)$. [Compare this with the result Q3a for the average number of visit to the origin]

(g) Use the result derived in Q3b, to give exact numerical value of the probability to return to origin in $d = 1, 2,$ and 3 dimension.

[This is the result first derived by Polya, and known as the Polya's theorem. This shows that RW is recurrent in 1d and 2d, means it will certainly come back to the origin. In $d \geq 3$ there is a non-zero theorem. This shows that RW is recurrent in 1d and 2d, means it will certainly come back to the origin. In $d \geq 3$, there is a non-zero probability for RW to escape without ever returning, ie., RW is transient. As RW $\longrightarrow$ Brownian Motion in the continuous limit, the same property holds for BM]

Question 4: number of distinct sites visited by RW & its similarity with prime numbers.

(a) Let N_n be the mean number of distinct sites visited by the RW in n number of steps. Show that [Explain why]

$$N_n = 1 + \sum_{\bar{\ell} \neq 0} \left[F_1(\bar{\ell}) + F_2(\bar{\ell}) + \cdots + F_n(\bar{\ell}) \right] \quad \cdots\cdots \quad (4)$$

(b) Guess the value of N_0 and N_1 without solving.

(c) Define $\Delta_n = N_n - N_{n-1}$, for $n \geq 1$. Then express N_n as a summation of $\Delta_k' s$ using N_0 and N_1.

(d) To derive Δ_n, use eq (4) to express Δ_n in terms of $F_n(\bar{\ell})$. Then, use this to write the relation between their generating functions

$$\Delta(z) = \sum_{n \geq 1}^{\infty} \Delta_n \cdot z^n \quad \text{and} \quad R(z,\bar{\ell}) = \sum_{n \geq 1} F_n(\bar{\ell}) \cdot z^n$$

(e) Use the solution in Eq(3), to show that

$$\Delta(z) = \frac{1}{(1-z)\, g(z,0)} - 1$$

[Hint: you need to use normalization $\sum_{\bar{\ell}} P_n(\bar{\ell}) = 1$]

(f) In Q3b you have derived the behavior of $g(z,0)$ near $z \to 1$ for all dimensions d. This gives how $\Delta(z)$ diverge for $z \to 1$, and this tells us how $\Delta_1 + \Delta_2 + \Delta_3 + \cdots + \Delta_n$ diverge for large n. To see this use the following result: writing $z = e^{-y}$ if $\Delta(e^{-y}) \sim \phi\left(\frac{1}{y}\right)$ for $y \to 0$, then

$$\Delta_1 + \Delta_2 + \cdots + \Delta_n \sim \phi(n), \text{ for large } n.$$

Use this to find the leading dependence of $\Delta_1 + \cdots + \Delta_n$ for large n, for all dimensions $d \geq 1$.

(g) Use the result derived in Q4c and show that

$$N_n \sim \begin{cases} \sqrt{n} & \text{for } d=1 \\ n/\log n & \text{for } d=2 \\ n & \text{for } d \geq 3 \end{cases}$$

(h) Knowing that for large n, the RW spread over a typical distance $\sqrt{n}$, comment on sparcity of the visited sites, in d-dimension.

(I) Relation to Prime numbers. Gauss showed that the number of primes less than n is $\simeq \dfrac{n}{\log n}$ which is same as N_n in 2d. This can be shown using sieve of Eratosthenes [read more on Wiki, if interested]. This is not a coincidence, and infact the two problem for large n can be related. [You may try to find more about this yourself, if interested]